MTH 605: Topology I

Homework V

(Due 16/10)

1. Show that any covering space of the torus T is isomorphic to T, or the plane, or $S^1 \times \mathbb{R}$.

2. Find all connected 2-sheeted, 3-sheeted, and 4-sheeted covering spaces of the figure 8 space $S^1 \vee S^1$ up to isomorphism.

3. Assume that $\pi_1(S^1 \vee S^1) \cong \langle a, b \rangle$, the free group on two letters a and b. Up to isomorphism, find the covering spaces of $S^1 \vee S^1$ that corresponds to the subgroups $\langle a \rangle \leq \pi_1(S^1 \vee S^1)$ and $\langle b \rangle \leq \pi_1(S^1 \vee S^1)$.

4. Given a group and a normal subgroup N, show that there exists a normal covering $\widetilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\widetilde{X}) \cong N$. and the deck transformation group $G(\widetilde{X} \to X) \approx G/N$.

5. Given covering actions of groups G_1 on X_1 and G_2 on X_2, show that the action of $G_1 \times G_2$ on $X_1 \times X_2$ defined by $(g_1, g_2)(x_1, x_2) = (g_1(x_1), g_2(x_2))$ is a covering space action, and that $(X_1 \times X_2)/(G_1 \times G_2) \approx X_1/G_1 \times X_2/G_2$.

6. For a covering space $p : \widetilde{X} \to X$ with X connected, locally path-connected, and semilocally simply-connected, show that

   (a) The components of $\widetilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

   (b) Under the Galois correspondence between connected covering spaces of X and the subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\widetilde{X}$ containing a given lift $\widetilde{x_0}$ of x_0 is the stabilizer of $\widetilde{x_0}$, the subgroup consisting of elements whose action on the fiber leaves $\widetilde{x_0}$ fixed.

7. Let X and Y be topological spaces. Suppose there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq i_Y$ and $g \circ f \simeq i_X$. Then X is said to be *homotopically equivalent* to Y (or X is said to have the *same homotopy type* as Y), denoted by $X \simeq Y$.

   (a) Please read Lemma 58.4, Corollaries 58.5-58.6, and Theorem 58.7 from Munkres.

   (b) Show that if a space X is contractible, then X has the same homotopy type as a point.

   (c) Show that if A is a deformation retract of X (see solution to Problem 6 of the MTH 605 midterm), then $X \simeq A$.

   (d) Give a counterexample to show that the converse of (b) does not hold always. [Hint: Consider $X = \cup_{n \in \mathbb{Z}_+} (1/n) \times I$ and $x_0 = (0, 1)$.]